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Some Remarks on H-Sets in Linear Approximation Theory

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In [1] Collatz presented a way to check the goodness of an approximating function, in relation with approximation by elements of a linear subspace V of the real Banach space $C(Q)$. The application of his inclusion theorem requires the choice of points in Q. However, they cannot be taken arbitrarily. For this reason Collatz introduced the concept of H-sets studied in [2, 3].

The primary purpose of this paper is to answer some remaining questions, mainly concerning invariance and construction principles for H-sets. Since the original definition of a minimal H-set is not well suited for our work, a dual definition is first derived (Section 1). The latter is accompanied by a dual inclusion theorem and it is shown (Section 2) that this generates sharper bounds than the estimates of [1]. A sufficient condition for invariance is deduced and turns out to be of great importance in our quest to understand the mechanism of invariance. This leads to practical rules of invariance under affine mappings, dilatations, and collineations (Section 3). Another question which arises rather naturally is how can H-sets be found and whether a given H-set relative to V can be constructed knowing that some subsets are H-sets relative to linear subspaces of V. These topics are briefly studied and it is shown under what restriction a relevant *decomposition* can be performed (Section 4). Based on this, we devise practical rules for *constructing* H-sets and minimal H-sets in a wide variety of situations. This constitutes the main contribution of this paper and is motivated by the fact that in general, H-sets prove to be quite difficult to obtain. Minimal H-sets relative to linear spaces generated by first-degree multivariate functions were characterized by Taylor in [7]. His results are complemented here by showing how minimal H-sets can be constructed from *blocks* and how these blocks can be deduced from H-sets (Section 5). It follows from this that there are recurrence relations between H-sets relative to different linear subspaces. Similar relations can be found in the construction rules for obtaining minimal H-sets relative to linear spaces generated by higher-degree multivariate functions (Section 6). Indeed, minimal H-sets relative to complicated generating families can be constructed from minimal H-sets relative to simpler

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ones. The latter minimal H-sets can be obtained, e.g., by block composition, but the main construction rule of Section 6 consists in combining an H-set and a B-set. The latter concept is a generalization of that of a block. This new technique is applicable to a wide variety of generating families. Moreover, for any given set of generating functions, this technique can be applied at the same time to several geometrical configurations. For example, in $\mathbb{R}^2$, minimal H-sets relative to the linear space generated by $\{1, x, y, x^2, xy, y^2\}$ were obtained by the above technique for about 500 geometrically different types of point sets (not to mention those merely obtained by affine or projective mappings which leave H-sets invariant).

1. H-Sets and Minimal H-Sets. Equivalent Definitions

We consider the linear hull V spanned by the family $\{\phi_1, ..., \phi_n\}$. By the original definition, a subset $D = \{P_1, ..., P_m\}$ of Q, is an H-set relative to V, if there exist m scalars ϵ_i ($\epsilon_i = +1$ or -1) such that for no $(a_1, ..., a_n) \in \mathbb{R}^n$, the following system of inequalities holds

$$\epsilon_i \cdot \sum_{j=1}^{n} a_j \cdot \phi_j(P_i) > 0, \qquad i = 1, ..., m.$$

Using matrix notation, the latter system can be written

$$\begin{bmatrix} \epsilon_1 & 0 & \cdots & 0 \\ \vdots & \epsilon_2 & & \vdots \\ 0 & \cdots & \cdots & \epsilon_n \end{bmatrix} \cdot \begin{bmatrix} \phi_1(P_1) & \cdots & \phi_n(P_1) \\ \vdots & & \vdots \\ \phi_1(P_m) & \cdots & \phi_n(P_m) \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} > 0_m . \tag{1}$$

Applying Gordan's theorem [5, p. 31] we have that either the system (1) $E \cdot \Phi \cdot a > 0_m$ has a solution, *or* $\Phi^T \cdot E \cdot \rho = 0_n$ has a nonzero solution $\rho = (\rho_1 \cdots \rho_m) \in \mathbb{R}^m$ where $\rho_i \geqslant 0$, $i = 1 \cdots m$, *but we never have both*. Consequently, if D is an H-set, there exist scalars ϵ_i, $i = 1 \cdots m$, such that (1) is inconsistent and by Gordan's theorem there exist positive scalars $\rho_1, ..., \rho_m$, not all zero, such that

$$\begin{bmatrix} \phi_1(P_1) & \cdots & \phi_1(P_m) \\ \vdots & & \vdots \\ \phi_n(P_1) & \cdots & \phi_n(P_m) \end{bmatrix} \cdot \begin{bmatrix} \epsilon_1\rho_1 \\ \vdots \\ \epsilon_m\rho_m \end{bmatrix} = 0_n . \tag{2}$$

Clearly the associated linear transformation $U_D \in \operatorname{Hom}(\mathbb{R}^m, \mathbb{R}^n)$

$$U_D : (\lambda_1, ..., \lambda_m) \to \left(\sum_{i=1}^{m} \lambda_i \phi_1(P_i), ..., \sum_{i=1}^{m} \lambda_i \Phi_n(P_i) \right) \tag{3}$$

cannot be a one-to-one mapping, which implies that $\ker U_D \neq \{\theta\}$ and hence the rank of the matrix $[\phi_i(P_j)]$, or equivalently $\operatorname{rk} U_D$, cannot be larger than $m - 1$. Conversely, if $\operatorname{rk} U_D \leq m - 1$, we have $\dim \ker U_D \geq 1$. For *any* nonzero element $(\lambda_1, ..., \lambda_m)$ in the kernel of U_D, (2) holds, taking $\epsilon_i = \operatorname{sign} \lambda_i$ and $\rho_i = |\lambda_i|$. Consequently, (1) has no solution and D is an H-set relative to V. We have thus proved that a dual definition of an H-set is as follows:

Definition. A set D is an H-set relative to V if the associated homomorphism U_D satisfies either $\operatorname{rk} U_D \leq (\operatorname{card} D) - 1$ or $\dim \ker U_D \geq 1$.

Since the rank of U_D cannot exceed $\min(m, n)$, it is immediately evident that point-sets D with a cardinal number larger than $n + 1$ are always H-sets. Collatz proposed a Gauss-like elimination method to verify whether or not a set D, with a given sign pattern, is an H-set relative to V. Here we only need to construct the kernel of the linear transformation U_D to obtain all possible sign patterns which can be associated with D so as to be an H-set. Both methods are dual to each other. The first is shorter but needs to be repeated several times, while the second is straightforward, computes all possibilities, and has further advantages to be shown in the next section. Since any subset of Q containing an H-set relative to V is also an H-set relative to V, it is immediately evident that minimal H-sets (which contain no proper subset which is an H-set relative to V) are of major importance. From the preceding we easily deduce the following dual definition of a minimal H-set.

Definition. An H-set D relative to V is *minimal* if the associated homomorphism U_D satisfies $\operatorname{rk} U_D = (\operatorname{card} D) - 1$, while the associated homomorphism $U_{D'}$ for every proper subset D' of D satisfies $\operatorname{rk} U_{D'} = \operatorname{card} D'$.

Clearly minimal H-sets relative to V have only two possible sign patterns and consist of no more than $n + 1$ points.

2. A Dual Inclusion Theorem

In this section we derive a dual version of the inclusion theorem. Its decisive advantage consists in producing better bounds for the distance between the function f, to be approximated, and V.

For the approximation of $f \in C(Q) \setminus V$ by elements of V, the inclusion theorem [2] guarantees the bound for the distance $d(f, V)$

$$\min\{\epsilon_i[f(P_i) - v_0(P_i)] : P_i \in D\} \leq d(f, V), \tag{4}$$

provided that D, with the ϵ_i, $i = 1, ..., m$, is an H-set relative to V, and the element $v_0 \in V$ satisfies

$$\epsilon_i[f(P_i) - v_0(P_i)] > 0 \qquad \text{for every} \quad P_i \in D. \tag{5}$$

In [4], Krabs generalized this theorem to approximation problems in normed linear spaces E. We wish to emphasize here the geometric interpretation. The calculation of a lower bound consists in constructing a family of hyperplanes $\{H_L \mid L \in \mathscr{L} \subset E^*\}$ such that there exists an element v_0 in V, common to all hyperplanes H_L, $L \in \mathscr{L}$, so that the linear subspace V is contained in the *union* of the half-spaces defined by these H_L, containing f in their complement. We then have

$$\inf\{d(f, H_L) \mid L \in \mathscr{L}\} \leqslant d(f, V).$$

We point out that even after $\mathscr{L}$ is determined, which, for $E = C(Q)$, corresponds to D, with the ϵ_i, being an H-set relative to V, there remains the difficulty of finding an element v_0 in V, common to all H_L. This difficulty does not arise in the following dual version which, in addition, is more straightforward and produces sharper estimates. Suppose $D \subset Q$ is an H-set relative to V, and $(\lambda_1, ..., \lambda_m) \in \ker U_D \backslash \{\theta\}$. Denote $\lambda_i = \epsilon_i \rho_i$, where $\rho_i \geqslant 0$. If we introduce the continuous linear functionals $L_i \in C(Q)^*$, defined by $L_i(f) = \epsilon_i f(P_i)$, $P_i \in D$, then as is known, each L_i is an extremal point of the dual unit ball in $C(Q)^*$. The linear functional $\mathfrak{L} = \sum_{i=1}^{m} \rho_i L_i$ satisfies $\mathfrak{L}(v) = 0$ for all $v \in V$. Consequently, we have $| \mathfrak{L}(f)| \leqslant \| \mathfrak{L} \| \cdot \| f - v \|$, and also

$$\frac{| \sum_{i=1}^{m} \lambda_i f(P_i)|}{\sum_{i=1}^{m} | \lambda_i |} \leqslant d(f, V). \tag{6}$$

Once the λ_i are known, the estimate (6) is obtained without determining an element $v_0 \in V$ for which (5) is valid. Moreover, the lower bound (6) for $d(f, V)$ is more accurate than (4), since

$$\min\{\epsilon_i[f(P_i) - v_0(P_i)] \mid P_i \in D\} \leqslant \frac{| \sum_{i=1}^{m} \lambda_i f(P_i)|}{\sum_{i=1}^{m} | \lambda_i |}.$$

Indeed, $\min\{d(f, H_i) \mid i = 1, ..., m\} \leqslant \sum_{i=1}^{m} \sigma_i d(f, H_i)$ for any $\sigma_i \geqslant 0$ with $\sum_{i=1}^{m} \sigma_i = 1$. The geometric interpretation of this dual inclusion theorem is as follows. Starting from D and $\lambda \in \ker U_D$, we construct a linear functional defining the homogeneous hyperplane $\mathscr{H} = H[\mathfrak{L}, 0]$. This $\mathscr{H}$ passes through V and satisfies $d(f, \mathscr{H}) \leqslant d(f, V)$. Moreover, the scalar $d(f, \mathscr{H})$ is a convex combination of the scalars $d(f, H_i)$, and hence larger than $\min\{d(f, H_i) \mid i = 1, ..., m\}$.

3. Invariance Principles

A question which frequently arises when studying H-sets is the following. Suppose a subset D of Q, with corresponding ϵ_i, is known to be an H-set relative to V. Are there other subsets of Q which are also H-sets, and whose sign patterns are strongly related to the original one? In this section we study the fundamental principles which govern invariance, and show how the generating family of V is constrained in several practical cases.

Consider one-to-one mappings T, which transform points of Q into points of Q. We want to characterize those mappings T which transform H-sets relative to V into H-sets relative to V in such a way that the set $\{T(P_1),..., T(P_m)\}$ will maintain at least all the sign patterns $(\epsilon_i \mid i = 1,..., m)$ of the H-set $D = \{P_1 ,..., P_m\}$. Such T are said to leave H-sets relative to V *invariant*.

PROPOSITION 1. *A one-to-one mapping T of Q into Q, leaves H-sets relative to V invariant if, for each $i = 1,..., n$, $\phi_i \circ T$ is in $V = \operatorname{span}\{\phi_1 ,..., \phi_n\}$.*

Indeed, if D is an H-set relative to V, there exists at least one nonzero element $(\lambda_1 ,..., \lambda_m)$ in $\ker U_D$. For such $(\lambda_1 ,..., \lambda_m)$ and for each $v \in V$, we have $\sum_{i=1}^{m} \lambda_i \cdot v(P_i) = 0$. If any $\phi_j \circ T$ is in V, we have $\sum_{i=1}^{m} \lambda_i \cdot \phi_j(T(P_i)) = 0$ for $j = 1,..., n$. Clearly $\{T(P_1),..., T(P_m)\}$ is then an H-set relative to V, and has the sign pattern $(\operatorname{sign} \lambda_i \mid i = 1,..., m)$ of D. Moreover, for this mapping T, we have $\ker U_D \subseteq \ker U_{T(D)}$.

The set of one-to-one mappings T of Q into Q which satisfy $\{\phi_i \circ T \mid i = 1,..., n\} \subset V$ will be denoted S. Clearly, S is a *semigroup* and the identity mapping I_Q is its unit. With each $T \in S$, we can associate the unique linear transformation $\tau \in \operatorname{End}(V)$ satisfying

$$v \circ T = \tau(v) \qquad \forall v \in V. \tag{7}$$

If to T_1, resp. T_2 corresponds τ_1, τ_2 in (7), then to the mapping $T_1 \circ T_2$ in S corresponds $\tau_2 \circ \tau_1$ in the ring $\operatorname{End}(V)$. The mappings T in S for which the linear transformation τ, defined by (7), is either injective or surjective, clearly satisfy $\ker U_D = \ker U_{T(D)}$, since $\{\phi_i \circ T \mid i = 1,..., n\}$ is then also a generating family of V. These injective or surjective elements in $\operatorname{End}(V)$ are necessarily automorphisms of V, elements of the linear group $\mathbf{GL}(V)$. In most practical cases S is a group of one-to-one mappings of Q onto Q, since it contains the symmetrical element T^{-1} for any mapping T in S. For these groups we now prove the following result.

PROPOSITION 2. *Given a group G of one-to-one mappings of Q onto Q, if for each $T \in G$ there is a unique element $\tau \in \operatorname{End}(V)$ such that $v \circ T = \tau(v)$ for every $v \in V$, then τ is necessarily an automorphism.*

This can be verified as follows. With T and T^{-1} in the group, τ and σ, respectively, are associated in $\operatorname{End}(V)$. Consequently, with $I_Q = T \circ T^{-1} = T^{-1} \circ T$ there is associated $1_V = \sigma \circ \tau = \tau \circ \sigma$ in $\operatorname{End}(V)$. Hence $\sigma = \tau^{-1} \in \operatorname{End} V$, and consequently, τ is in the linear group $\mathbf{GL}(V)$. Q.E.D.

We conclude that the mappings T of the group G (of Proposition 2) leave H-sets relative to V invariant and that they satisfy $\ker U_D = \ker U_{T(D)}$.

In the following, Q is the affine space $\mathbb{R}^S$ with the coordinate system $0; x, y, \dots, z$. Well-known groups of one-to-one mappings on $\mathbb{R}^S$ onto $\mathbb{R}^S$ are $\mathbf{H}(S, \mathbb{R})$, the group of all translations and nonzero scalar multiplications, the affine group $\mathbf{GA}(S, \mathbb{R})$ of all one-to-one affine mappings, the linear group $\mathbf{GL}(S, \mathbb{R})$ of all automorphisms of $\mathbb{R}^S$, and the projective group $\mathbf{PGL}(S + 1, \mathbb{R})$ of the projective space $\mathbf{P}(\mathbb{R}^{S+1})$, containing all collineations. It is important to note that in considering such mappings with respect to invariance, we are naturally led to look for affine and projective properties as guides for the construction of H-sets. Thus, collinearity of points and concurrency of lines will form the framework for the H-sets considered. In most practical examples we consider V spanned by the family $\{(x^i y^j \cdots z^k) \mid (i, j, \dots, k) \in K\}$. The question which arises is under which condition on K, H-sets relative to V are invariant under mappings of the above-mentioned groups. Therefore, we examine under which condition on K, a mapping τ can be constructed, for a given T, according to (7).

PROPOSITION 3. *H-sets relative to* $V = \operatorname{span}\{(x^i y^j \cdots z^k) \mid (i, j, \dots, k) \in K\}$ *are invariant under mappings in* $\mathbf{H}(S, \mathbb{R})$ *if* $[(0, 1, \dots, a) \times (0, 1, \dots, b) \times \cdots \times (0, 1, \dots, c)] \subset K$ *whenever* $(a, b, \dots, c) \in K$.

To verify this proposition, we consider $V = \operatorname{span}\{(x^i y^j) \mid (i, j) \in K\}$. For any $P = (\xi, \eta) \in \mathbb{R}^2$, the coordinates of $T(P)$ are $\xi' = \lambda\xi + t_1$ and $\eta' = \lambda\eta + t_2$, where $\lambda \neq 0$. For any $(a, b) \in K$ we can write $[x^a y^b] \cdot (T(P)) = \sum_{i=0}^{a} \sum_{j=0}^{b} t_{ij} \cdot [x^a y^b] \cdot (P)$, where the scalars t_{ij} are uniquely determined. Hence the endomorphism τ associated with T by (7) exists whenever $(a, b) \in K$ implies $[(0, 1, \dots, a)] \times [(0, 1, \dots, b)] \subset K$. For example, H-sets relative to $V = \{(x^i y^j) \mid (i, j) \in K_1\}$, where K_1 is given in Fig. 1a, are invariant under the mappings in $\mathbf{H}(2, \mathbb{R})$.

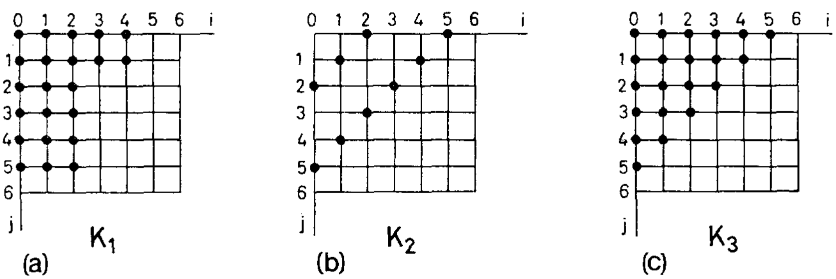

FIGURE 1

PROPOSITION 4. *H-sets relative to* $V = \operatorname{span}\{(x^i y^j \cdots z^k) \mid (i, j, \dots, z) \in K\}$ *are invariant under linear transformations in* $\mathbf{GL}(S, \mathbb{R})$ *if* $\{(i, j, \dots, k) \mid i + j + \cdots + k = a + b + \cdots + c;\ i, j, \dots, k \in [0, 1, \dots, a + b + \cdots + c]\} \subset K$ *whenever* $(a, b, \dots, c) \in K$.

For any $P = (\xi, \eta) \in R^2$, we consider $T(P)$ with coordinates $\xi' = \alpha_{11}\xi + \alpha_{12}\eta$ and $\eta' = \alpha_{21}\xi + \alpha_{22}\eta$, where $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \neq 0$. For any $(a, b) \in K$ there exist uniquely defined scalars t_{ij} such that $[x^a y^b] \cdot (T(P)) = \sum_{i=0}^{a} \sum_{j=0}^{b} t_{ij}[x^{a+b-(i+j)} y^{i+j}](P)$. Clearly the endomorphism τ associated with T by (7) exists if whenever $(a, b) \in K$, we have that $\{(c, d) \mid c + d = a + b,\ c, d \in [0, 1,..., a + b]\} \subset K$. To illustrate this, we have that H-sets relative to $V = \text{span}\{(x^i y^j) \mid (ij) \in K_2\}$, where K_2 is given in Fig. 1b, are invariant under mappings in $\mathbf{GL}(2, \mathbb{R})$. From Propositions 3 and 4, we immediately obtain

PROPOSITION 5. *H-sets relative to V are invariant under affine mappings of $\mathbf{GA}(S, \mathbb{R})$ as well as under collineations of $\mathbf{PGL}(S + 1, \mathbb{R})$ if*

$$\{(i, j,..., k) \mid i + j + \cdots + k = a + b + \cdots + c;\quad i, j,..., k \in [0, 1,..., a + b + \cdots + c]\} \subset K,$$

whenever $(a, b,..., c) \in K$.

For example, H-sets relative to $V = \text{span}\{(x^i y^j) \mid (ij) \in K_3\}$, where K_3 is given in Fig. 1c, are invariant under mappings in $\mathbf{GA}(2, \mathbb{R})$ as well as in $\mathbf{PGL}(3, \mathbb{R})$.

4. DECOMPOSITION PRINCIPLES FOR H-SETS

In the present section we briefly analyze how a particular block structure of the matrix $M(U_D)$, corresponding to the homomorphism (3), leads to decomposition principles for H-sets. We first study an H-set relative to V and its subset, an H-set relative to a proper subspace of V. Second, we show an H-set broken down into several subsets, called blocks.

Let a set D be given. We suppose the matrix $M(U_D)$ to have the structure

$$M(U_D) = \begin{bmatrix} M(U_{11}) & M(U_{12}) \\ 0 & M(U_{22}) \end{bmatrix} \tag{8}$$

where $U_{11} \in \text{Hom}(\mathbb{R}^{m_1}, \mathbb{R}^{n_1})$, $U_{12} \in \text{Hom}(\mathbb{R}^{m_2}, \mathbb{R}^{n_1})$, and $U_{22} \in \text{Hom}(\mathbb{R}^{m_2}, \mathbb{R}^{n_2})$. The present analysis is limited to (8), but the results can easily be extended to $\alpha \times \alpha$ diagonal block structures.

If D is an H-set and $(\lambda_1, \lambda_2) \in \ker U_D \backslash \{\theta\}$, then $\lambda_2 \in \ker U_{22}$ and $\lambda_1 \in \mathbb{R}^{m_1}$ solve the system $U_{11}(\lambda_1) = -U_{12}(\lambda_2)$. Excluding trivial cases, we have $\text{rk}\, U_{22} \leqslant m_2 - 1$, and the set $D_2 = \{P_{m_1+1},..., P_m\}$ is then an H-set relative to the linear space V_2 spanned by $\{\phi_{n_1+1},..., \phi_n\}$. By the hypothesis on D, we have that $U_{12}(\ker U_{22}) \subset (\ker {}^t U_{11})^0$ [8, p. 85]. It is important to note that the signs $\{\epsilon_i \mid i = 1,..., m_1\}$ of the points of $D_1 = \{P_1,..., P_{m_1}\}$ are dependent of those of D_2. Consequently, no general rule can be given for the sign pattern of D_1 in the H-set $D_1 \cup D_2$, relative to $V = \text{span}\{\phi_1,..., \phi_n\}$.

DEFINITION. The set D_1 is called a *B-set* relative to $\{\phi_1, ..., \phi_{n_1}\}$ and with respect to D_2, an *H*-set relative to $\{\phi_{n_1+1}, ..., \phi_n\}$, if $U_{12}(\ker U_{22}) \subset [\ker({}^tU_{11})]^0$, provided that $M(U_{D_1 \cup D_2})$ has the structure (8).

A particular form of $M(U_{12})$ which occurs in applications where $\phi_1(P) = 1$ for any $P \in D$, is

$$M(U_{12}) = \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n_1 \cdot m_2}. \tag{9}$$

Due to the inherent symmetry of the problem, it is convenient to change the notation. We consider the set $D = D_1 \cup D_2$ which is supposed to be a (minimal) *H*-set relative to $V = \mathrm{span}\{1, \psi_1, ..., \psi_{n_1}, \chi_1, ..., \chi_{n_2}\}$; card $D_1 = m_1$, card $D_2 = m_2$. The structure of the matrix $M(U_D)$ is

$$M(U_D) = \begin{bmatrix} 1 \ \cdots \ 1 & 1 \ \cdots \ 1 \\ M(U_1) & 0 \\ 0 & M(U_2) \end{bmatrix} \tag{10}$$

provided that $U_1 \in \mathrm{Hom}(\mathbb{R}^{m_1}, \mathbb{R}^{n_1})$, and $U_2 \in \mathrm{Hom}(\mathbb{R}^{m_2}, \mathbb{R}^{n_2})$. Excluding trivial cases, we obtain immediately that D_1 is a (minimal) *H*-set relative to $V_1 = \mathrm{span}(\psi_1, ..., \psi_{n_1})$, and D_2 is a (minimal) *H*-set relative to $V_2 = \mathrm{span}(\chi_1, ..., \chi_{n_2})$. Moreover, by the preceding analysis, D_1 is a *B*-set relative to $\{1, \psi_1, ..., \psi_{n_1}\}$, and so is D_2 (interchanging the roles of D_1 and D_2). However, D_1 is a very particular *B*-set, since its sign pattern is only globally influenced by that of the *H*-set D_2, in the sense that $(\lambda_1, \lambda_2) \in \ker U_D$ is equivalent to $\lambda_1 \in \ker U_1$, $\lambda_2 \in \ker U_2$, and $\sum_{i=1}^{m_1} \lambda_{1i} = -\sum_{j=1}^{m_2} \lambda_{2j}$. D_1 and D_2 are therefore called blocks. It was Collatz who introduced in [2, pp. 50–52] the concept of block for points on a line. Clearly our definition is a generalization of the original. We conclude that the (minimal) *H*-set $D_1 \cup D_2$ relative to $V = \mathrm{span}\{1, \psi_1, ..., \psi_{n_1}, \chi_1, ..., \chi_{n_2}\}$, for which $M(U_D)$ has the form (10), can be decomposed into two blocks: D_1 relative to $\{\psi_1, ..., \psi_{n_1}\}$, and D_2 relative to $\{\chi_1, ..., \chi_{n_2}\}$.

DEFINITION. A set $B = \{P_1, ..., P_m\}$ is called a *Block* relative to $\{\phi_1, ..., \phi_n\}$ if $\phi_i \not\equiv 1$ for $i = 1, ..., n$, and if

$$\mathrm{rk}[\phi_i(P_j) \mid i = 1, ..., n, j = 1, ..., m] \leqslant m - 1.$$

DEFINITION. The block B, relative to $\{\phi_1, ..., \phi_n\}$ is *minimal* if $\mathrm{rk}[\phi_i(P_j) \mid i = 1, ..., n, j = 1, ..., m] = m - 1$, and each of its submatrices of m' columns has rank m'.

The block D_1 together with (sign $\lambda_{1i} \mid i = 1, ..., n_1$), where $\lambda_1 \in \ker U_1$, is said to be a positive block, if $\sum_{i=1}^{m_1} \lambda_{1i} > 0$. The (minimal) H-set D is consequently decomposed into a "positive" and a "negative" block.

DEFINITION. A set $B = \{P_1, ..., P_m\}$ together with $(\epsilon_i \mid i = 1, ..., m)$ forms a *positive block* relative to $\{\phi_1, ..., \phi_n\}$, if B is an H-set relative to span$\{\phi_1, ..., \phi_n\}$, and if $\epsilon_i = \text{sign}\,\lambda_i$ for $i = 1, ..., m$, provided that $(\lambda_1, ..., \lambda_m) \in \ker U_B$, where $\sum_{i=1}^{m} \lambda_i > 0$.

Multiple block decomposition can be similarly treated. The signs of the blocks can be freely chosen, provided that there is at least one positive and one negative block. However multiple block composition can never lead to minimal H-sets.

In most practical cases, blocks can easily be obtained. The following proposition is readily verified.

PROPOSITION 6. *If D is a subset of an affine space, and if*

$$\psi_i(\theta) = 0, \qquad i = 1, ..., m, \tag{11}$$

then D is a (minimal) block relative to $\{\psi_1, ..., \psi_m\}$ iff $\{\theta\} \cup D$ is a (minimal) H-set relative to span$\{1, \psi_1, ..., \psi_m\}$. *Moreover, if $(\epsilon_0, \epsilon_1, ..., \epsilon_m)$ is a sign pattern of the H-set $\{\theta\} \cup D$, then the positive block D has the signs $(\epsilon_i \mid i = 1, ..., m)$ if $\epsilon_0 = -1$, and $(-\epsilon_i \mid i = 1, ..., m)$ if $\epsilon_0 = 1$.*

It follows that the blocks introduced by Collatz in [2] are obtained from minimal H-sets relative to span$\{1, x, x^2, ..., x^n\}$. It is known that such minimal H-sets consist of $n + 2$ points on the x-axis, with alternating signs. For example, for $n = 3$, Fig. 2a shows a minimal H-set relative to span$\{1, x, x^2, x^3\}$ with a positive block (Fig. 2c) and negative blocks (Figs. 2b,d), all relative to $\{x, x^2, x^3\}$. Finally we remark that whenever a minimal H-set $\{\theta\} \cup D$ relative to $V = \text{span}\{1, \psi_1, ..., \psi_m\}$ is given, D is a block relative to $\{\psi_1, ..., \psi_m\}$, which is itself a minimal H-set relative to span$\{\psi_1, ..., \psi_m\}$. Thus the sets of Figs. 2b,c,d are minimal H-sets relative to span $\{x, x^2, x^3\}$.

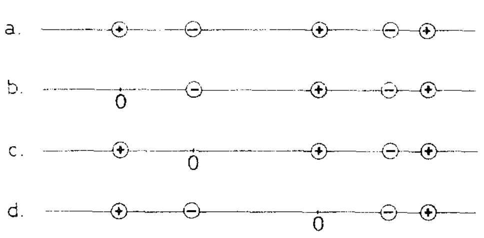

FIGURE 2

In the remaining sections, the above decomposition will be applied in the construction of more complicated H-sets.

5. MINIMAL H-SETS FOR FIRST-DEGREE MULTIVARIABLE FUNCTIONS

In [7], Taylor examined and described all minimal H-sets relative to the linear space of first-degree multivariable functions, interpreting the original definition of H-sets. We briefly derive the main results, taking advantage of the dual definition, and applying the foregoing block decomposition. It will be shown that all minimal H-sets are composed of a positive and a negative block, and, moreover, that there is only one type of block for the class in question of generating families.

We consider here V generated by the family $\{1, x_1, \ldots, x_S\}$, and D, a subset of an affine space $Q = \mathbb{R}^S$, with a coordinate system $0; x_1, \ldots, x_S$. If we suppose D and $(\epsilon_i \mid i = 1, \ldots, m)$ to be an H-set, then we can consider D as the union of D_+ and D_- corresponding to the points of D for which $\epsilon_i = 1$, resp. $\epsilon_i = -1$. This together with the particular form of the generating family implies that there does not exist a hyperplane in $\mathbb{R}^S$ separating D_+ and D_-. As an immediate consequence we have that the intersection $\text{co}(D_+) \cap \text{co}(D_-)$ is nonvoid; for, otherwise, there would exist a hyperplane separating the convex hulls and, hence, also D_+ and D_-. If, moreover, D is a minimal H-set, then the decomposition of D into D_+ and D_- is unique. By the particular structure of the generating family, we have that $\bar{\lambda} \in \ker U_D$, $\sum_{i=1}^{m} |\bar{\lambda}_i| = 1$, is equivalent to the existence of a unique $\xi \in \mathbb{R}^S$ such that $\xi \in \text{co}(D_+) \cap \text{co}(D_-)$. We thus obtain the important result that the convex hull of D_+ (resp. D_-) is contained in an $m_1 - 1$ (resp. $m_2 - 1$)-dimensional linear space, where $m_1 = \text{card}\, D_+$, and $m_2 = \text{card}\, D_-$. Finally, by Proposition 5, we know that H-sets relative to $\text{span}\{1, x_1, \ldots, x_S\}$ are invariant under affine and projective transformations, and, consequently, we need only examine the particular situation: $D = D_1 \cup D_2$, where

$$
\begin{gathered}
0 \in \text{co}(D_1) \cap \text{co}(D_2), \qquad \text{co}(D_i) \subset \mathbb{R}^{m_i}, \qquad i = 1, 2, \\
\mathbb{R}^S = \mathbb{R}^{m_1 - 1} \oplus \mathbb{R}^{m_2 - 1}.
\end{gathered} \tag{12}
$$

The analysis of this case is straightforward since, by (12), the homomorphism U_D has the block structure (10). The minimal H-set D is, consequently, decomposed into a positive block D_1, and a negative block D_2, or conversely. The positive block D_1 is obtained from the minimal H-set $\{\theta\} \cup D_1$ relative to $\text{span}\{1, x_1, \ldots, x_{m_1 - 1}\}$ by omitting one point which must be the origin (Proposition 6). This H-set can be geometrically described by the fact that the origin lies in the convex hull of the set $D_1 \subset \mathbb{R}^{m_1 - 1}$. Moreover, every subset of $m_1 - 1$ vectors of $\{\mathbf{OP}_i \mid i = 1, \ldots, m_1\}$ forms a free family. The sign of the origin in the H-set $\{\theta\} \cup D_1$ is negative, and hence the opposite of those of the points of the block D_1 which is supposed to be a positive block. Here all points of the positive block have a positive sign. Conversely, it is easily verified that two such blocks D_1 and D_2 relative to $\{x_1, \ldots, x_{m_1 - 1}\}$, resp.

to $\{x_{m_1}, ..., x_{m_1+m_2-2}\}$, are obtained from minimal H-sets relative to span$\{1, x_1, ..., x_{m_1-1}\}$, resp. span $\{1, x_{m_1}, ..., x_{m_1+m_2-2}\}$, such that $M(U_D)$ has the structure (10), and form a minimal H-set relative to the linear space generated by $\{1, x_1, ..., x_{m_1+m_2-2}\}$ in $\mathbb{R}^S$, $(s = m_1 + m_2 - 2)$. For the minimal H-sets D relative to $V = \text{span}\{1, x_1, ..., x_S\}$, and with reference to (12), we can determine explicitly the elements of the kernel of U_D. Indeed, if $\mathbf{OO} = \sum_{i=1}^{m_1} \alpha_i \cdot \mathbf{OP}_i = \sum_{j=1}^{m_2} \beta_j \cdot \mathbf{OP}_{j+m_1}$, where $\alpha_i > 0$ and $\beta_j > 0$ with $\sum_{i=1}^{m_1} \alpha_i = \sum_{j=1}^{m_2} \beta_j = 1$ are unique, then we have that the elements of ker U_D are multiples, either of (i) $(\bar{\lambda}_1, ..., \bar{\lambda}_{m_1+m_2})$ where $\bar{\lambda}_i = \alpha_i/2$ for $i = 1, ..., m_1$, corresponding to the points of the positive block D_1, and $\bar{\lambda}_{m_1+j} = \beta_j/2$, $j = 1, ..., m_2$, corresponding to the points of the negative block D_2, or of (ii) $(-\bar{\lambda}_1, ..., -\bar{\lambda}_m)$ where D_1 is a negative block, and D_2 a positive block.

6. Minimal H-Sets for Higher-Degree Multivariable Functions

We deduce now some rules for constructing minimal H-sets relative to linear spaces generated by higher-degree multivariable functions. The following techniques are applicable in a wide variety of cases and are all based on recurrence-like relations between minimal H-sets. Indeed, by resorting to Section 4, complicated H-sets can be constructed either from simpler ones, from an H-set and a B-set, or by multiple bloc composition.

For given generating families, H-sets and minimal H-sets can be obtained if H-sets are known for some related generating families. Consider an arbitrary set D, and let $\{\phi_0 \cdot \phi_i \mid i = 1, ..., n\}$ be the generating family of V, where we assume that $\phi_0(P) \neq 0$ for all $P \in D$. One can easily obtain information on H-sets relative to V. Suppose the set $D = \{P_1, ..., P_m\}$ with $(\epsilon_i \mid i = 1, ..., m)$ is an H-set relative to the linear space $\mathscr{V}' = \text{span}\{\phi_1, ..., \phi_n\}$. It is easily verified by (1) or (2) that the set D with $(\epsilon_i \cdot \text{sign}\ \phi_0(P_i) \mid i = 1, ..., m)$ is an H-set relative to V. Moreover, if U_D (resp. $\mathscr{U}_D$) is the homomorphism associated with D relative to V (resp. $\mathscr{V}'$), then for any $(\lambda_1, ..., \lambda_m) \in \ker \mathscr{U}_D$, we have $(\lambda_1/\phi_0(P_1), ..., \lambda_m/\phi_0(P_m)) \in \ker U_D$. Finally, if D is a minimal H-set relative to $\mathscr{V}'$, then, obviuosly, D is also a minimal H-set relative to V. To illustrate this result, we first refer to the set $D = \{P_1, ..., P_4\}$ in the system 0; x, y, which, together with the signs as in Fig. 3, forms a minimal H-set

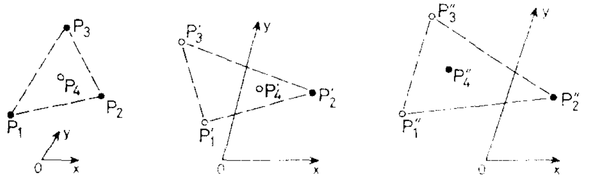

Figure 3

relative to span$\{1, x, y\}$ (cf. Section 5). Such an H-set is known to be invariant under affine or projective transformation. Consequently, the sets $\{P_1',..., P_4'\}$ and $\{P_1'',..., P_4''\}$ are minimal H-sets relative to span$\{x^a y^b, x^{a+1} y^b, x^a y^{b+1}\}$, where a is an odd number and b arbitrary.

We describe now H-sets relative to V, which can be constructed knowing an H-set relative to a linear subspace of V, and a B-set. Consider a set $D = D_1 \cup D_2$, and a linear space V generated by the family $\{\psi_1,..., \psi_{n_1}, \phi_0\phi_1,..., \phi_0\phi_{n_2}\}$, where $\phi_0(P) = 0$ for all $P \in D_1$. Clearly the matrix $M(U_D)$ has the block structure (8). For D to be an H-set relative to V, it is necessary that D_2 be an H-set relative to $V_2 = \text{span}\{\phi_0\phi_1,..., \phi_0\phi_{n_2}\}$. We suppose now that D_2 is a minimal H-set relative to span$\{\phi_1,..., \phi_{n_2}\}$, and $\phi_0(P) \neq 0$ on D_2. Assume, moreover, that the points of D_1 are obtained from a minimal H-set of $n_1 - 1$ points, relative to span$\{\psi_1,..., \psi_{n_1}\}$, by deleting one point. Under these assumptions we have rk $U_{11} = m_1 = n_1$, rk $U_{22} = m_2 - 1 \leqslant n_2$, and rk $U_D \leqslant m - 1 \leqslant n$. Consequently D_1 is a B-set relative to span$\{\psi_1,..., \psi_{n_1}\}$ and with respect to the H-set D_2. The essential conclusion we can draw from this is that the set D constructed this way is a minimal H-set relative to V. Indeed, rk $U_D = m - 1$, and any subset $\mathscr{D} \subset D$, with card $\mathscr{D} < m$, also satisfies rk $U_{\mathscr{D}} = \text{card}\, \mathscr{D}$.

To illustrate this technique for constructing H-sets relative to higher-degree multivariable functions, we consider the generating family $\{1, x, y, x^2, xy, y^2\}$ spanning V. H-sets relative to V are known to be invariant under affine transformations. Consider $D = \{P_1,..., P_7\}$ in Fig. 4a. We can apply the preceding technique, taking $V_1 = \text{span}\{1, x, x^2\}$ and $V_2 = \text{span}\{y, yx, y^2\}$, $D_1 = \{P_5, P_6, P_7\}$, $D_2 = \{P_1,..., P_4\}$. All hypotheses are fulfilled and, consequently, D is a minimal H-set relative to V. Moreover, we can determine the sign pattern of D_2 (cf. Section 5). However, the signs of the B-set D_1 cannot easily be obtained. This shortcoming can be circumvented, by taking at least two points of D_1 on lines joining points of D_2, so as to create subsets of D consisting of three collinear points. Due to the invariance under affine transformation, we can determine the signs of the remaining points. Indeed,

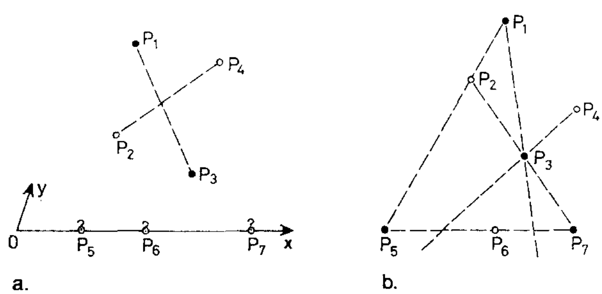

FIGURE 4

in the configuration of Fig. 4b, we take first the x-axis along P_1P_5. We can determine the signs of P_6 and P_7 in a similar way to the above, considering the set $D_2' = \{P_3, P_4, P_6, P_7\}$. Repeating this to determine the signs of the remaining point P_5, we take the x-axis along P_2P_7. Thus, we completely determine the signs of the minimal H-set D relative to V. What is important here is that no four points are collinear unless the H-set reduces to the set of these points, with alternating signs.

The above technique is applicable to many configurations. For example, the method just used can be applied in about 500 geometrically different cases, and leads to minimal H-sets in $0; x, y$, relative to $V = \text{span}\{1, x, y, x^2, xy, y^2\}$. Two cases are presented in Figs. 5a, 5b.

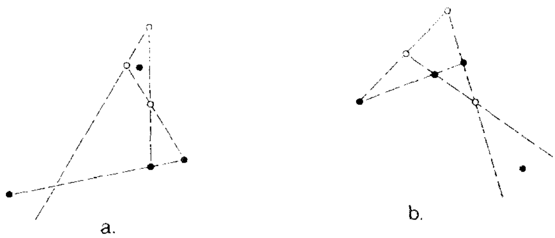

FIGURE 5

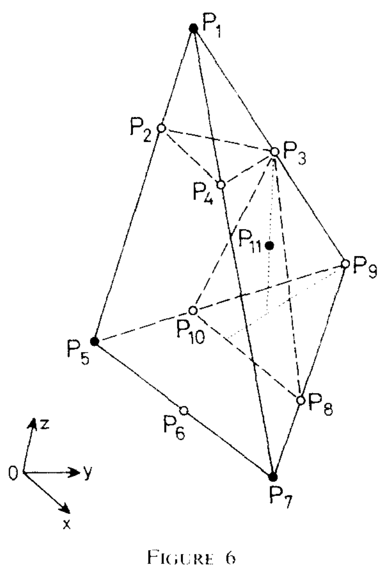

FIGURE 6

This technique is applicable to a wide variety of generating families. An example is given in Fig. 6, and represents a minimal H-set relative to $\text{span}\{1, x, y, z, x^2, y^2, z^2, xy, yz, zx\}$. Taking first $0; x, y$ in the plane determined by P_5, P_7, and P_9, we determine the signs of the points of $D_2 = \{P_1, P_2, P_3, P_4, P_{11}\}$ relative to span$\{1, x, y, z\}$. Taking successively 0; x, y in the plane determined by P_1, P_5, P_2 and in the plane determined by P_1, P_7, P_9, we find the signs of P_8, P_9, P_{10}, and P_6.

These examples make it clear that our technique is applicable in a wide variety of cases, building H-sets from simpler ones. This establishes a kind of recurrence between H-sets. Indeed, the examples of Figs. 4, 5, and 6 were constructed on the basis of the H-sets of Section 5, but the newly obtained H-sets can again be used to construct more complicated ones. To illustrate this recurrence, we have determined the signs of the configurations shown in Fig. 7. They are all minimal H-sets relative to span$\{1, x, y\} = V_1$ (in Fig. 7a), $V_1 \oplus$ span$\{x^2, xy, y^2\} = V_2$ (in Fig. 7b), $V_2 \oplus$ span$\{x^3, x^2y, xy^2, y^3\} = V_3$ (in Fig. 7c), and $V_3 \oplus$ span$\{x^4, x^3y, x^2y^2, xy^3, y^4\}$ (in Fig. 7d).

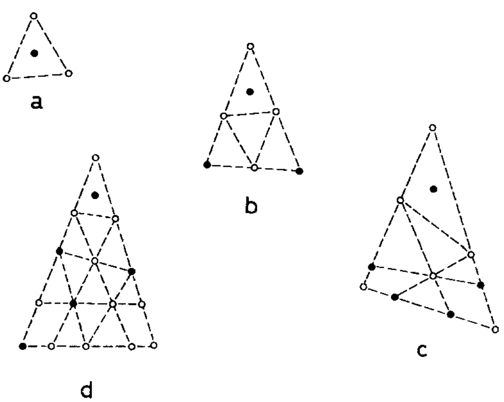

FIGURE 7

Another technique for obtaining minimal H-sets consists in block composition. This method is most suitable for obtaining H-sets for generating families which contain higher-degree ($\geqslant 4$) multivariable functions. Let the set D_1, consisting of $n_1 + 1$ points, be a minimal block relative to $\{\psi_1, ..., \psi_{n_1}\}$, and D_2 a minimal block relative to span$\{\chi_1, ..., \chi_{n_2}\}$. If, moreover, no ψ_i, $i = 1, ..., n_1$, or χ_j, $j = 1, ..., n_2$, is constant, and if $\psi_i(P) = 0$ for all $P \in D_2$ and $i = 1, ..., n_1$; $\chi_j(P) = 0, \forall P \in D_1$, $j = 1, ..., n_2$, then the set $D = D_1 \cup D_2$ is a minimal H-set relative to $V =$ span$\{1, \psi_1, ..., \psi_{n_1}, \chi_1, ..., \chi_{n_2}\}$. If Proposition 6 can be applied to one of these generating families, then the blocks can be obtained from the H-set $\{\theta\} \cup D_1$ relative to span$\{1, \psi_1, ..., \psi_{n_1}\}$, or the H-set $\{\theta\} \cup D_2$ relative to span$\{1, \chi_1, ..., \chi_{n_2}\}$.

To illustrate, we consider the family

$$\{1, x, y, z, x^2, y^2, z^2, xy, yz, xz, x^3, y^3, z^3, x^2y, xy^2, y^2z, yz^2, x^2z, xz^2, xyz\}. \quad (13)$$

Figures 8a,c show two H-sets relative to the linear space generated by (13). In Fig. 8a, the H-set consists of a positive block (Fig. 2c) and two negative blocks (Figs. 2b,d), and is obtained by an affine transformation. In Fig. 8b, we represent the minimal H-set relative to span$\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$ from which the negative block making up the H-set of Fig. 8c is derived. The latter is clearly a minimal H-set relative to the linear space generated by (13).

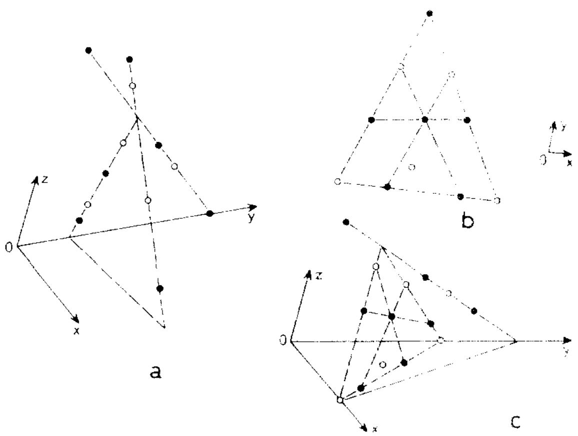

FIGURE 8

Finally, it is interesting to recall here that given an H-set $\mathscr{S}$ relative to V, $\mathscr{S}$ is also an H-set relative to any linear subspace V' of V. Moreover, if $\{\theta\} \cup D$ is a minimal H-set relative to span$\{1, \phi_1, ..., \phi_n\}$, and if $\phi_i(\theta) = 0$ for $i = 1, ..., n$, then by Section 4, D is a minimal H-set relative to the linear space spanned by the family $\{\phi_1, ..., \phi_n\}$. For example, D of Fig. 9 (in $\mathbb{R}^3$) is a

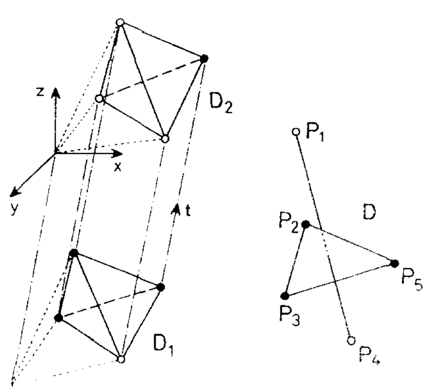

FIGURE 9

minimal H-set relative to span$\{1, x, y, z\}$. Consequently D_1 and D_2 in Fig. 9 are minimal H-sets relative to span$\{x, y, z\}$. They are invariant under linear transformations ($\in \mathbf{GL}(3, \mathbb{R})$). Clearly D_2 is obtained by translating D_1. However translations do not leave these H-sets invariant, which is evident here.

For completeness, here are two more construction techniques, useful in practice.

Let $V = \text{span}\{x^i y^j \mid (i,j) \in K\}$ and $q = \max\{i + j \mid (i,j) \in K\}$. A set consisting of $2q + 2$ points (in the affine space 0; x, y), situated on a sheet of a conic section and with alternating signs, forms an H-set relative to V. A proof is given in [2, p. 54]. In Fig. 10, a minimal H-set relative to $V = \text{span}\{1, x, y, z, x^2, y^2, z^2, xy, yz, xz\}$ is shown; here a block is obtained from a minimal H-set on a conic section.

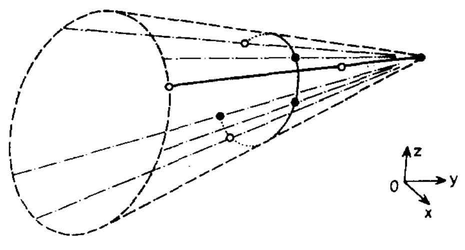

FIGURE 10

The other technique to obtain H-sets is an immediate consequence of the Euler–Jacobi theorem. This is discussed by Shapiro in [6]. For a particular V, the kernel of U_D is known and is related to the Jacobian determinant

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